Covering a Chessboard with Staircase Walks

Eyal Ackerman* Rom Pinchasi†

July 22, 2013

Abstract

An *ascending* (resp., *descending*) *staircase walk* on a chessboard is a rook's path that goes either right or up (resp., down) in each step. We show that the minimum number of staircase walks that together visit every square of an $n \times n$ chessboard is $\lceil \frac{2}{3}n \rceil$.

1 Introduction

The motivation to this paper was a question raised by Lapid Harel, an undergraduate student in a course taught by the second author in the Technion in 2012. He asked the following question.

Problem A. *What is the minimum number of lines that intersect the interior of every square of an $n \times n$ chessboard?*

It is clear that n lines suffice and it is not hard to see that $n/2 + 1$ lines are necessary, as each line can intersect the interior of at most $2n - 1$ squares. Where exactly the truth is in between, is still open.

Here we consider Problem A for curves instead of lines, such that every curve is a graph of a strictly increasing or strictly decreasing function (lines clearly satisfy this property). We may assume without loss of generality that no curve intersects a corner of a square, since otherwise we can shift it a little and extend the set of squares whose interior it intersects. Therefore, the squares whose interior a curve intersects form a *staircase walk* on the chessboard.

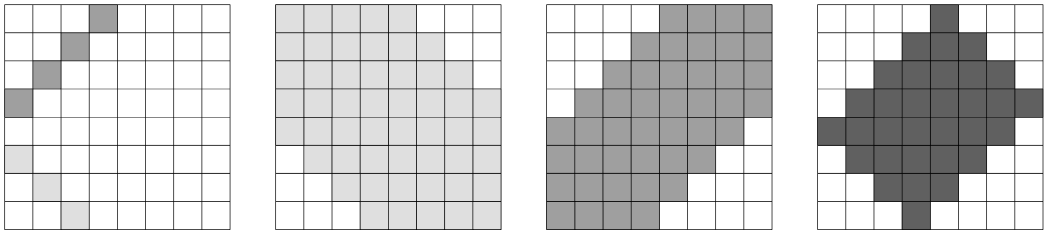
Definition 1 (Staircase walk)**.** An *ascending* (resp., *descending*) *staircase walk* is a rook's path on a chessboard that goes either right or up (resp., down) in every step.

For the purely combinatorial question of finding the minimum number of staircase walks that cover an entire $n \times n$ chessboard we were able to find the exact answer.

Theorem 1. *The minimum number of staircase walks that together visit each square of an $n \times n$ chessboard is $\lceil \frac{2}{3}n \rceil$.*

*Department of Mathematics, Physics, and Computer Science, University of Haifa at Oranim, Tivon 36006, Israel. ackerman@sci.haifa.ac.il.

†Mathematics Department, Technion—Israel Institute of Technology, Haifa 32000, Israel. room@math.technion.ac.il. Supported by BSF grant (grant No. 2008290) and by ISF grant (grant No. 1357/12).



(a) D_3 and C_4 (C_4 is darker). (b) B^3 is shaded (c) B_4 is shaded (d) B_4^3 is shaded

Figure 1: Illustrations of the terms used in the proof of the lower bound.

In Section 2 we prove that $\lceil \frac{2}{3}n \rceil$ staircase walks are always needed, while in Section 3 we give a construction showing that this bound is tight.

Theorem 1 clearly gives a lower bound for Problem A. However, it easy to see that not every staircase walk can be "realized" by a line. For example, one cannot draw a line that intersects the squares of a walk consisting of the first row and first column of an $n \times n$ chessboard, for $n > 2$. Another example is the construction in Section 3 (otherwise, we would have settled Problem A).

Related work. We are not aware of any work that studies the problems that are described above, or similar ones. However, there is a vast literature on enumeration of lattice paths satisfying various restrictions (including being staircase walks). See, e.g., [1] and [2].

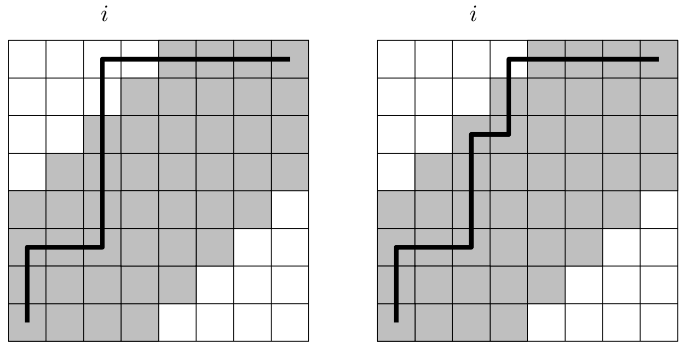
2 The lower bound

In this section we show that at least $\lceil \frac{2}{3}n \rceil$ staircase walks are needed to cover an $n \times n$ chessboard. We denote by (i, j) the square in the i'th column and the j'th row. Therefore, $(1, 1)$ denotes the bottom-left square and (n, n) denotes the top-right square. Notice that without loss of generality we may assume that all ascending staircase walks start at $(1, 1)$ and end at (n, n) (or else we can extend them to be such). Similarly, we may assume that all descending staircase walks start at $(1, n)$ and end at $(n, 1)$. We continue with a few definitions and some notation.

Definition 2. For every $i = 1, \ldots, 2n-1$ we denote by D_i the i'th *descending diagonal* of the $n \times n$ chessboard. That is, D_i is the set of all squares at position (x, y) such that $x + y = i + 1$. Similarly, for every $i = 1, \ldots, 2n-1$ we denote by C_i the i'th *ascending diagonal* of the $n \times n$ chessboard. That is, C_i is the set of all squares at position (x, y) such that $x - y = i - n$. See Figure 1(a) for an example of these terms.

We denote by $B = B_0$ the entire $n \times n$ chessboard. For every $i > 0$ we define $B_i = \bigcup_{j=i+1}^{2n-(i-1)} C_j$ and $B^i = \bigcup_{j=i+1}^{2n-(i-1)} D_j$. In other words, B_i (resp., B^i) is the board without the first and last i descending (resp., ascending) diagonals. We denote by B_j^i the intersection $B^i \cap B_j$. See Figure 1 for examples of these terms.

We say that two walks are *disjoint* if they do not share a common square. The next lemma is crucial for the proof. It will imply that if we have p ascending staircase walks and q descending staircase walks, then we can assume that the ascending (resp., descending) walks lie inside B_q (resp., B^p) and are disjoint within B_q^p.



(a) ℓ_1 (b) ℓ'_1

Figure 2: Modifying an ascending staircase walk to be in B_q.

Lemma 1. *Let $\ell_1, \ldots, \ell_p$ be p ascending staircase walks and let $q \leq n - p$. Then there exist p ascending staircase walks $\ell'_1, \ldots, \ell'_p$, such that:*
(1) $\ell'_1, \ldots, \ell'_p$ are contained in B_q;
(2) $\ell'_1, \ldots, \ell'_p$ are disjoint in $D_p \cup D_{p+1} \cup \ldots \cup D_{2n-p}$;
(3) $\ell'_1, \ldots, \ell'_p$ cover the first and last $(p-1)$ descending diagonals; and
(4) $\ell'_1, \ldots, \ell'_p$ cover all the squares in B_q that are covered by $\ell_1, \ldots, \ell_p$.

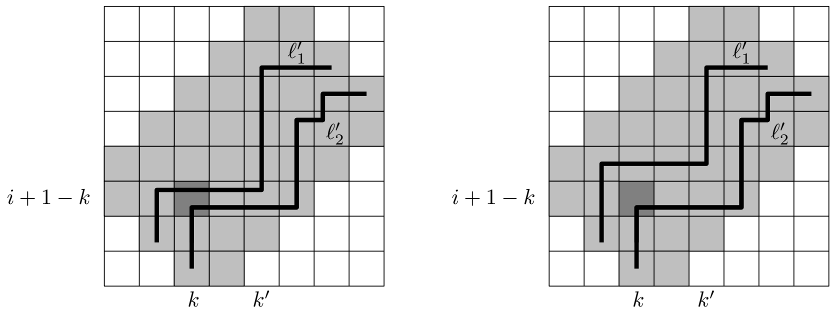
Proof. We first show how to modify the walks, such that they will be contained in B_q. Suppose for example that ℓ_1 is not contained in B_q. Then without loss of generality ℓ_1 contains a square from C_q (the other possible case is symmetric, namely when ℓ_1 contains a square on C_{2n-q}). Let i be the smallest integer such that the square $(i, n-q+i)$ of diagonal C_q is included in ℓ_1. It must be that the square just below it, namely $(i, n-q+i-1)$, is also included in ℓ_1 because of the minimality of i. Let j be the smallest integer such that $(i+1, j) \in \ell_1$. We must have that $j \geq n-q+i$ because $(i, n-q+i) \in \ell_1$ and ℓ_1 is an ascending staircase walk. We now modify ℓ_1 by removing the squares $(i, n-q+i), \ldots, (i, j)$ from ℓ_1 and adding the squares $(i+1, n-q+i), (i+1, n-q+i+1), \ldots, (i+1, j-1)$ (see Figure 2 for an example of these steps). The resulting walk ℓ'_1 contains all the squares in $B_q \cap \ell_1$ and now the smallest index i' such that the square $(i', n-q+i')$ of diagonal C_q is included in ℓ'_1 is at least $i+1$, if at all exists. Therefore, after at most q such steps we will end up with a modified ℓ_1 that does not contain any square on C_q.

Let $\ell'_1, \ldots, \ell'_p$ be the modified walks that are contained in B_q. Next we chop the curves by removing from every curve its intersection with the first and last $(p-1)$ descending diagonals $D_1, \ldots, D_{p-1}$ and $D_{2n-p+1}, \ldots, D_{2n-1}$.

Suppose that $\ell'_1, \ldots, \ell'_p$ are not disjoint within $\bigcup_{j=p}^{2n-p} D_j$, and let i be the smallest index such that D_i contains a square that belongs to two walks. Since every ascending staircase walk contains exactly one square from each descending diagonal and $|D_i \cap B_q| \geq p$ it follows that there is at least one square in $D_i \cup B_q$ that is not covered by any of the walks $\ell'_1, \ldots, \ell'_p$.

Let j and k be two indices such that $(j, i+1-j) \in B_q^p$ is not covered by $\ell'_1, \ldots, \ell'_p$ and $(k, i+1-k)$ is covered by at least two paths from $\ell'_1, \ldots, \ell'_p$ and $|j-k|$ is the smallest. Without loss of generality we can assume that $j < k$, or else we can flip the chessboard about the main diagonal C_n. We may also assume that ℓ'_1 and ℓ'_2 are two walks that contain $(k, i+1-k)$.

Let $(k', i+1-k+1)$ be the leftmost square on the $(i+1-k+1)$'th row of B that is contained in ℓ_1 or ℓ_2. Without loss of generality we assume that ℓ'_1 contains this square. Notice that $k' \geq k$ because



(a) before (b) after

Figure 3: Modifying ℓ'_1 and ℓ'_2.

ℓ'_1 and ℓ'_2 are ascending walks. It follows that the squares $(k, i+1-k), (k+1, i+1-k), \ldots, (k', i+1-k)$ are all contained in both ℓ'_1 and ℓ'_2. There are two cases to consider.

Case 1. $i > p$. We modify ℓ'_1 and ℓ'_2 in the following way. Because of the minimality of i it follows that the squares $(k-1, i+1-k)$ and $(k, i-k)$ are contained in ℓ'_1 and ℓ'_2.

Suppose that ℓ'_1 contains $(k-1, i+1-k)$ and ℓ'_2 contains $(k, i-k)$. In this case we modify ℓ'_1 from the point $(k-1, i+1-k)$ to go one square up to $(k-1, i+2-k)$ and then all the way to the right until square $(k', i+2-k)$, and then continue along the original path of ℓ'_1. See Figure 3 for an example. If ℓ'_2 contains $(k-1, i+1-k)$ and ℓ'_1 contains $(k, i-k)$, then we can just switch the "tails" of ℓ'_1 and ℓ'_2 (ℓ'_1 will follow ℓ'_2 until the square $(k, p+1-k)$, and vice versa), and then we have the previous case.

Case 2. $i = p$. In this case both ℓ'_1 and ℓ'_2 start at $(k, p+1-k)$, since we previously chopped the walks. We modify ℓ'_1 in the following way. We let ℓ'_1 be the ascending walk that starts at $(k-1, p+2-k)$, goes all the way right to $(k', p+2-k)$ and then follows the previous walk of ℓ'_1.

Note that by these modifications (either in Case 1 or in Case 2) we can only add squares of B covered by $\ell'_1, \ldots, \ell'_p$ that are part of B_q. Moreover, after these modifications either there is one more square on D_i that is covered by the union of $\ell'_1, \ldots, \ell'_p$ (this happens if $k = j+1$), or $(k-1, p+2-k)$ is covered by at least two paths from $\ell'_1, \ldots, \ell'_p$ and therefore we can repeat this step with a smaller value of $|j-k|$ (or at least we have reduced the number of such pairs j, k, if there were more than one pair with the minimum absolute difference). Hence after finitely many such steps every square on D_i is either covered by a unique walk from $\ell'_1, \ldots, \ell'_p$, or it it not covered at all. In particular every square in D_p (resp, D_{2n-p}) is covered by exactly one walk.

To complete the proof of the lemma note that it is very easy to extend the walks $\ell'_1, \ldots, \ell'_p$ so that they will cover all the squares on the diagonals $D_1, \ldots, D_{p-1}$ and $D_{2n-p+1}, \ldots, D_{2n-1}$ without changing the situation on the diagonals $D_p, \ldots, D_{2n-p}$. We illustrate this for the diagonals $D_1, \ldots, D_{p-1}$ and a symmetric argument applies for the diagonals $D_{2n-p+1}, \ldots, D_{2n-1}$. Without loss of generality we assume that square $(i, p+1-i)$ belongs to ℓ'_i for $i = 1, \ldots, p$. For every $1 \leq i \leq p$ we modify ℓ'_i by starting at $(1,1)$, then going right all the way to $(i, 1)$, then up all the way to $(i, p+1-i)$, and then continuing along ℓ'_i. This way we cover all the squares on diagonals $D_1, \ldots, D_{p-1}$, without changing the situation on the squares of B_q^p. □

By reflecting the chessboard about a horizontal line we can deduce from Lemma 1 the following analogous lemma:

Lemma 2. *Let $\ell_1, \ldots, \ell_q$ be q descending staircase walks and let $p \leq n - q$. Then there exist q descending staircase walks $\ell'_1, \ldots, \ell'_q$, such that:*
(1) $\ell'_1, \ldots, \ell'_q$ are contained in B_p;
(2) $\ell'_1, \ldots, \ell'_q$ are disjoint in $C_q \cup C_{q+1} \cup \ldots \cup C_{2n-q}$;
(3) $\ell'_1, \ldots, \ell'_q$ cover the first and last $(q-1)$ ascending diagonals; and
(4) $\ell'_1, \ldots, \ell'_q$ cover all the squares in B_p that are covered by $\ell_1, \ldots, \ell_q$.

We are now ready to prove Theorem 1. Suppose we can cover the entire $n \times n$ chessboard B by p ascending staircase walks and q descending staircase walks. We aim to show that $p + q \geq \lceil \frac{2}{3}n \rceil$. Therefore, we can clearly assume that $p + q \leq n$. Using Lemmas 1 and 2, we can assume that the ascending walks are contained in B_q and the descending walks are contained in B^p. Moreover, we can assume that no two ascending walks share a common square in B^p_q and no two descending walks share a common square in B^p_q.

The number of squares in B^p_q is equal to $n^2 - p(p+1) - q(q+1)$. Every ascending walk contains precisely $2n-1-2p$ squares from B^p_q. Similarly, every descending walk contains precisely $2n-1-2q$ squares from B^p_q. The important observation is that every ascending walk and every descending walk must share at least one common square. This square must be located in B^p_q because the ascending walks are contained in B_q while the descending walks are contained in B^p.

We conclude that the number of squares in B^p_q which is $n^2 - p(p+1) - q(q+1)$ must be smaller than or equal to $p(2n-1-2p) + q(2n-1-2q) - pq$ which is the total number of squares covered by the ascending and descending walks in B^p_q minus at least pq distinct times where the same square in B^p_q is covered by an ascending walk and a descending walk. Those squares are distinct because no two ascending walks share a square in B^p_q and the same is true for descending walks.

Therefore,

$$n^2 - p(p+1) - q(q+1) \leq p(2n-1-2p) + q(2n-1-2q) - pq.$$

After some easy manipulations we obtain

$$(n - (p+q))^2 \leq pq.$$

The right hand side is always smaller than or equal to $\left(\frac{p+q}{2}\right)^2$ and therefore,

$$(n - (p+q))^2 \leq \left(\frac{p+q}{2}\right)^2,$$

from which we conclude that $p + q \geq \frac{2}{3}n$. Since $p + q$ is an integer, we have that $p + q \geq \lceil \frac{2}{3}n \rceil$. □

3 The upper bound

In this section we show that it is always possible to cover an $n \times n$ chessboard with $\lceil \frac{2}{3}n \rceil$ staircase walks.

It is easy to see that a 3×3 chessboard can be covered by one ascending walk and one descending walk (for obvious reasons we omit a figure). Given any $3k \times 3k$ chessboard, we can cover it with k ascending walks and k descending walks as follows (see Figure 4 for an example). Let $\ell_1, \ldots, \ell_k$

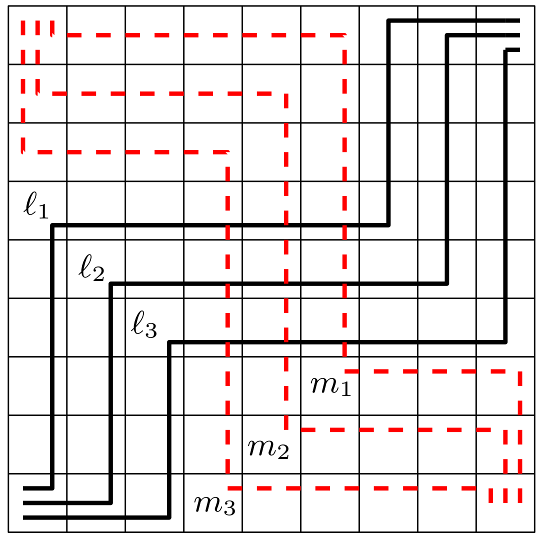


Figure 4: Covering a 9×9 board with 3 ascending walks and 3 descending walks.

be the following ascending walks. For every $1 \leq i \leq k$ let ℓ_i start from $(1,1)$, then go right all the way to $(i,1)$, then go all the way up to $(i, 2k-i+1)$, then right all the way to $(2k+i, 2k-i+1)$, then up all the way to $(2k+i, 3k)$, and then right all the way to $(3k, 3k)$.

Let $m_1, \ldots, m_k$ be the following descending walks. For every $1 \leq i \leq k$ let m_i start from $(1, 3k)$, then go down all the way to $(1, 3k-i+1)$, then go all the way right to $(2k-i+1, 3k-i+1)$, then down all the way to $(2k-i+1, k-i+1)$, then right all the way to $(n, k-i+1)$, and finally down all the way to $(n, 1)$.

It is easy to check by inspection that the $2k$ staircase walks $\ell_1, \ldots, \ell_k$ and $m_1, \ldots, m_k$ cover the entire $3k \times 3k$ chessboard.

Therefore, we can cover a $3k \times 3k$ chessboard by $2k$ staircase walks. If we are given a $(3k+1) \times (3k+1)$ board, then we can cover the top row and right column by one descending walk and the remaining $3k \times 3k$ board by $2k$ walks as before. Similarly, if we are given a $(3k+2) \times (3k+2)$ board, then we can cover the two top rows and two rightmost columns by two descending walks, and the remaining $3k \times 3k$ board by $2k$ walks as before. $\square$

References

[1] S.G. Mohanty, *Lattice Path Counting and Applications*, New York: Academic Press, 1979.

[2] T.V. Narayana, *Lattice Path Combinatorics with Statistical Applications*, Toronto, Ontario, Canada: University of Toronto Press, 1979.