On coloring points with respect to rectangles

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Abstract

In a coloring of a set of points P with respect to a family of geometric regions one requires that in every region containing at least two points from P, not all the points are of the same color. Perhaps the most notorious open case is coloring of n points in the plane with respect to axis-parallel rectangles, for which it is known that $O(n^{0.368})$ colors always suffice, and $\Omega(\log n/\log^2 \log n)$ colors are sometimes necessary.

In this note we give a simple proof showing that every set P of n points in the plane can be colored with $O(\log n)$ colors such that every axis-parallel rectangle that contains at least *three* points from P is non-monochromatic.

1 Introduction

A *hypergraph* (or *range space*) H consists of a vertex set V and a (hyper)edge set $E \subseteq 2^V$. A (valid) *coloring* of H assigns a color to every vertex in H such that in every edge of H that contains at least two vertices not all vertices are of the same color (that is, the edge is *non-monochromatic*). If in every edge there is a color that appears (at least once and) at most k times, then we say that the coloring is *k-conflict-free* (1-conflict-free coloring is called *conflict-free* coloring).

Colorings and conflict-free colorings of hypergraphs that stem from geometric regions have attracted some attention lately, due to their applications to frequency assignment in wireless networks, scheduling in RFID networks, and decompositions of multiple coverings (see the recent survey of Smorodinsky on conflict-free coloring [11] and the references therein). In a typical geometric setting, the vertex set of the hypergraph is a set of points P, and its edge set is defined by a family of regions $\mathcal{F}$, such that every region $F \in \mathcal{F}$ defines an edge that consists of the points of P that belong to F.

Perhaps the most challenging open question concerning coloring of geometric hypergraphs is to find tight asymptotic bounds for the minimum number of colors that suffice for coloring any set P of n points in the plane such that any axis-parallel rectangle that contains at least two points from P contains points of different colors. Har-Peled and Smorodinsky [8] were the first to consider this problem and gave an $O(\sqrt{n})$ upper bound. Soon after, several others pointed out that this can be slightly improved to $O(\sqrt{n/\log n})$ [8, 9]. Ajwani et al. [1] significantly improved the bound to $O(n^{0.382})$, and very recently Chan [6] obtained the currently best upper bound of $O(n^{0.368})$. A lower bound of $\Omega(\log n/\log^2 \log n)$ was proved by Chen et al. [7], and it was conjectured that the upper bound should also be polylog(n). In this note we give a simple proof for such an upper bound when considering only rectangles that contain at least *three* points. In fact, in the following theorem we prove a more general result that relates to the notion of *k-colorful* coloring, introduced in [3]. In what follows and throughout the rest of the paper log stands for $\log_2$.

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Theorem 1. *For every integer $k > 0$, every set P of n points in the plane can be colored with $O(k^4 \log n)$ colors such that every axis-parallel rectangle that contains at least $2k-1$ points from P contains points of at least k different colors.*

Next we describe two consequences of Theorem 1. For a set of points P and an integer $d > 1$, let $\alpha_d(P)$ be the maximum size of a *d-independent* subset $Q \subseteq P$, where Q is d-independent if there is no axis-parallel rectangle R such that there are exactly d points from P in R and all of them belong to Q. Chen et al. [7] proved that if P is a set of n points that is randomly and uniformly selected from the unit square, then almost surely $\Omega(n/\log^{1/(d-1)} n) \le \alpha_d(P) \le O(dn \log^2 \log n/\log^{1/(d-1)} n)$. Theorem 1 implies that $\alpha_3(P) = \Omega(n/\log n)$ for every set P of n points (by coloring P with $O(\log n)$ colors and taking the largest color class). This bound can be further improved using a result of Alon; see a remark at the end of Section 2.

Corollary 2. *For every set P of n points $\alpha_3(P) \ge c_1 \frac{n \log\log n}{\log n}$, where $c_1 > 0$ is an absolute constant.*

Note that this bound is quite close to the above-mentioned upper bound $\alpha_3(P) \le O(\frac{n \log^2 \log n}{\sqrt{\log n}})$ due to Chen et al. [7]. Their upper bound also implies that the upper bound in Theorem 1 cannot be $o(\frac{\sqrt{\log n}}{\log^2 \log n})$.

Clearly the bound of Corollary 2 holds for $\alpha_d(P)$ when $d \ge 3$, since any d-independent subset is also d'-independent, for every $d' \ge d$. It would be interesting to show a bound that increases with d.

Corollary 3. *For every integer $d \ge 3$, every set of n points in the plane can be $(d-1)$-conflict-free colored with respect to axis-parallel rectangles using $c_2 \frac{\log^2 n}{\log\log n}$ colors, where $c_2 > 0$ is an absolute constant.*

Proof. We prove the claim by induction on n. We may assume that n is large enough by choosing $c_2 > 0$ big enough. Define $P_0 = P$. For every $i \ge 0$ and as long as $|P_i| \ge \frac{n}{2}$, apply Corollary 2 for the set P_i and find a d-independent set in P_i of size at least $c_1 \frac{(n/2) \log\log(n/2)}{\log(n/2)} \ge c_1 \frac{(n/2) \log\log n}{\log n}$. Color this set by a new color and remove it from P_i to obtain P_{i+1}.

When we stop $i = t$ and we have $|P_t| \le \frac{n}{2}$. We used t colors and we observe that $t \le \frac{n}{2}/(c_1 \frac{(n/2) \log\log n}{\log n}) + 1 \le \frac{4}{c_1} \frac{\log n}{\log\log n}$. We use the induction hypothesis and color the remaining points in P_t by at most $c \frac{\log^2(n/2)}{\log\log(n/2)} \le c \frac{(\log n - \frac{1}{2})^2}{\log\log n}$ colors. Now we just need to choose $c > 0$ big enough such that $c \frac{(\log n - \frac{1}{2})^2}{\log\log n} + \frac{4}{c_1} \frac{\log n}{\log\log n} \le c \frac{\log^2 n}{\log\log n}$, which is definitely possible.

Smorodinsky [11] showed that this scheme of finding a d-independent subset, coloring it with a new color, removing it, and continuing it the same manner indeed yields a $(d-1)$-conflict-free colroing. □

2 Proof of Theorem 1

Let k be a positive integer, and let P be a set of n points in the plane. We may assume that $k \ge 2$, since the theorem trivially holds for $k = 1$. A *q-rectangle* is an axis-parallel rectangle that contains exactly q points from P. A rectangle is *k-colorful* if it contains points of at least k different colors. Since any q-rectangle such that $q > (2k-1)$ contains a $(2k-1)$-rectangle, it is enough to prove that P can be colored with $O(k^4 \log n)$ colors such that every $(2k-1)$-rectangle is k-colorful.

We may assume without loss of generality that no two points in P share the same x- or y-coordinate, since otherwise a slight perturbation of the points can only extend the set of $(2k-1)$-rectangles. Moreover, only the relative position of the points rather than their actual coordinates matters when it comes to the set of $(2k-1)$-rectangles they induce. Thus, we may assume that the points in P lie on an $n \times n$ portion of the integer grid.

For an integer i we define the graph $G_i(P)$ as follows. The vertices of $G_i(P)$ are the points of P. Two points $p, q \in P$ form an edge in $G_i(P)$ if there exists a k-rectangle that contains both of them and whose aspect ratio is $1/2^i$ (the aspect ratio of a rectangle is the ratio between its height and its width). For a set I of integers let $G_I(P) = \bigcup_{i \in I} G_i(P)$.

Lemma 2.1. *$G_I(P)$ has $O(k^4|I|n)$ edges.*

Proof. Rectangles of the same aspect ratio intersect as pseudo-disks, that is, the boundaries of two rectangles intersect at most twice. By [5, Thm. 13] the number of combinatorially different pseudo-disks containing exactly k points from a planar set of n points is $O(k^2 n)$. Therefore, there are at most $O(k^2 n)$ combinatorially different k-rectangles of a given aspect ratio. Since we consider $|I|$ different aspect ratios and every k-rectangle induces $O(k^2)$ edges, the number of edges of $G_I(P)$ is $O(k^4|I|n)$. □

Recall that a graph is *c-colorable* if it is possible to assign to each of its vertices one of c colors, such that different colors are assigned to adjacent vertices. From Lemma 2.1 we can conclude:

Lemma 2.2. *$G_I(P)$ is $O(k^4|I|)$-colorable.*

Proof. By lemma 2.1 there exists a constant c such that $G_I(P)$ has at most $ck^4|I|n$ edges. It is easy to show by induction on n that $G_I(P)$ is $(ck^4|I| + 1)$-colorable. For $n \leq k$ the claim trivially holds. Let P be a set of $n > k$ points. Since $G_I(P)$ contains at most $ck^4|I|n$ edges, it has a vertex v of degree at most $ck^4|I|$. Let $P' = P \setminus \{v\}$. By the induction hypothesis $G_I(P')$ is $(ck^4|I| + 1)$-colorable. If two points $p, q \in P'$ are both in a rectangle R of aspect ratio x that contains exactly k points from P, then there is also a rectangle of aspect ratio x that contains them both and $k - 2$ other points from P': It could be R if $v \notin R$ or a rectangle obtained from R by extending it while maintaining its aspect ratio until hitting a new point (there is such a point since $n > k$). Therefore, $G_I(P) \setminus \{v\}$ is a subgraph of $G_I(P')$ and hence $G_I(P) \setminus \{v\}$ is $(ck^4|I| + 1)$-colorable. Since v has at most $ck^4|I|$ neighbors, the $(ck^4|I| + 1)$-coloring of $G_I(P) \setminus \{v\}$ can be extended to a $(ck^4|I| + 1)$-coloring of $G_I(P)$. □

With Lemma 2.2 in hand we can now complete the proof of Theorem 1. Consider $G = G_I(P)$ for $I = \{-\lfloor \log n \rfloor, \ldots, \lceil \log n \rceil\}$. By Lemma 2.2, there is a coloring of G with $O(k^4 \log n)$ colors. We will show that under this coloring every $(2k-1)$-rectangle is k-colorful.

Let R be a $(2k-1)$-rectangle, let h and w be its height and width, respectively, and assume without loss of generality that $h \leq w$. Since we assume that the points in P lie on an $n \times n$ grid, we have $h \geq 1$ and $w \leq n$. Therefore, there is an integer $1 \leq t \leq \lceil \log n \rceil$ such that $2^{t-1}h \leq w \leq 2^t h$. Thus, we can cover R (and no point in $\mathbb{R}^2 \setminus R$) by two, possibly overlapping, rectangles of height h and width $2^{t-1}h$. One of these rectangles contains at least k points of P. If it contains more points, then we can shrink it continuously while maintaining its aspect ratio until it contains exactly k points of P. The k points inside the rectangle (whose aspect ratio is $1/2^{t-1}$) form a k-clique in G, and therefore are colored by k different colors. □

Remarks

- For the case $k = 2$ it is not necessary to use [5, Thm. 13] for the proof of Lemma 2.1, since it is not hard to show that $G_i(P)$ is planar in this case (see, e.g., [5, Thm. 2]). Moreover, it is also easy to see that for every vertex v in G the subgraph $G[N(v)]$ is 4-colorable, where $N(v)$ denotes the neighbors of v and $G[U]$ denotes the subgraph induced by a vertex subset U. (The Southeast neighbors of v can be colored alternately with two colors. The same two colors can be used to color the Northwest neighbors of v. Two additional colors can be used to color the Southwest and Northeast neighbors of v.) Alon [2] proved that if G is a graph with average degree d and $G[N(v)]$ is r-colorable for every vertex v, then

G has an independent set of size at least $c_r \frac{n \log d}{d}$, where c_r depends only on r. It follows that $G_I(P)$ has an independent set of size $\Omega(\frac{n \log\log n}{\log n})$, which implies Corollary 2.

- The approach that was used in the proof of Theorem 1 will not work in three and higher dimensions. For example, Figure 1 shows a set of n points in three dimensions that determines $\Omega(n^2)$ axis-parallel cubes that contain exactly two points.

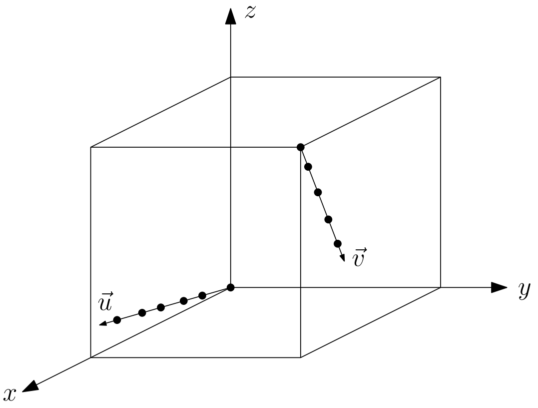

Figure 1: A set of n points that determines $\Omega(n^2)$ axis-parallel cubes that contain exactly two points ($\vec{u}$ lies on the xy plane and $\vec{v}$ is parallel to the yz plane).

- Bar-Noy et al. [4] gave a general framework for *online*[1] conflict-free coloring *k-degenerate* hypergraphs. Their approach and the proof of Theorem 1 imply:

  Corollary 4. *There is a randomized algorithm that online-colors every set P of n points in the plane with respect to axis-parallel rectangles containing at least three points from P and uses $O(\log^2 n)$ colors with high probability.*

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