There are not too many Magic Configurations

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Abstract

A finite planar point set P is called a *magic configuration* if there is an assignment of positive weights to the points of P such that, for every line l determined by P, the sum of the weights of all points of P on l equals 1. We prove a conjecture of Murty from 1971 and show that a magic configuration consists either of points in general position, or all points are collinear, with the possible exception of one point, or they form a special configuration of 7 points.

1 Introduction

Let P be a finite set of points in the plane. P is called a *magic configuration* if there is an assignment of positive weights to the points of P such that, for every line l determined by P, the sum of the weights of all points of P on l equals 1. Figure 1 shows an example of a point set that is a magic configuration. This special point set (and any projective transformation of it) is called a *failed Fano* configuration. We prove a conjecture of Murty [Mu71] saying that apart from failed Fano configurations, every set of n points that is a magic configuration is either in general position, or contains $n-1$ collinear points. A few other remarks on the history of the problem can be found in *The Open Problems Project* [DMO].

Theorem 1. *A magic configurations of cardinality n is either*

- *a configuration with $n-1$ (or n) collinear points, or*
- *a configuration in general position, that is, with no three points on a line, or*
- *a configuration with 7 points that up to a projective transformation is depicted in Figure 1.*

We will now make some preliminary observations regarding magic configurations. Many of these observations can be found already in Murty's paper [Mu71].

Assume that a configuration P of $n \geq 2$ points in the plane is magic and that its points are assigned positive weights that witness the fact that P is magic. Recall that a line determined by P is called *ordinary* if it includes precisely two points of P. By the Gallai-Sylvester theorem [Ga44, Sy93], the points of P must determine an ordinary line unless they are all collinear.

We claim that unless P has $n-1$ collinear points, then for every point $p \in P$ there is an ordinary line not passing through p. Assume otherwise. By the theorem of Kelly and Moser [KM58], the set

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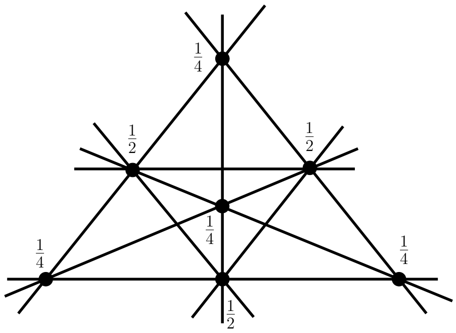


Figure 1: *Failed Fano* configuration

$P\setminus\{p\}$ determines at least $\frac{3}{7}(n-1)$ ordinary lines (see [CS93] for the current best bound on the number of ordinary lines determined by n points). By our assumption, all these lines must pass through p. It follows that at most $\frac{1}{7}(n-1)$ points of $P\setminus\{p\}$ lie on an ordinary line through p and these are all the ordinary lines determined by P, contradicting the Kelly-Moser theorem.

It is now easy to see that unless P has $n-1$ collinear points, every point through which there is an ordinary line must be assigned the weight $\frac{1}{2}$. To see this assume that p is such a point and assume without loss of generality that it is assigned a weight that is greater than $\frac{1}{2}$. (Otherwise look at the other point on the ordinary line through p.) Let q and r be two points different from p that constitute an ordinary line in P. One of q and r is assigned a weight greater than or equal to $\frac{1}{2}$. The sum of the weights assigned to the points on the line through that point and p will be strictly greater than 1, a contradiction.

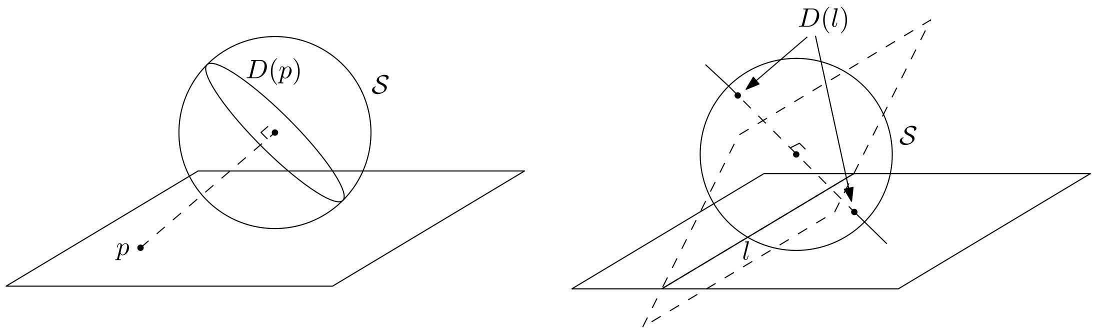
Denote by A the set of all points in P through which there is an ordinary line, and assume that P does not have $n-1$ collinear points. Then each point in A is assigned a weight of $\frac{1}{2}$. It follows that any line through two points in A must be ordinary. Denote by B the set $P\setminus A$. Clearly, every point in B must be assigned a weight that is strictly smaller than $\frac{1}{2}$. Indeed, let $b\in B$ and $a\in A$. The line through a and b cannot be ordinary for otherwise $b\in A$. Therefore it must contain a third point c. As the weight assigned to a is $\frac{1}{2}$, it follows that the weight assigned to b can be at most $\frac{1}{2}$ minus the weight assigned to c.

Theorem 1 will therefore follow from the following theorem.

Theorem 2. *Let A and B be two nonempty sets of distinct points in the Euclidean plane. Assume that all the ordinary lines determined by $A\cup B$ are precisely all the lines determined by two points of A. Assume further that every point in $A\cup B$ is assigned a positive weight such that the sum of the weights of all points on any given line determined by $A\cup B$ is 1. Then the configuration of points $A\cup B$ is a failed Fano configuration that is equal, up to a projective transformation, to the one shown in Figure 1, where A consists of the points whose weight is $\frac{1}{2}$.*

Instead of proving Theorem 2 we will prove its dual theorem on the unit sphere $\mathcal{S}$ that touches the plane at the origin. We refer here to the standard duality under which the dual of a point in the plane is a great circle on $\mathcal{S}$, and the dual of a line is a pair of antipodal points on $\mathcal{S}$. For a point p in the plane, the dual $D(p)$ is the great circle on $\mathcal{S}$ which is the intersection of $\mathcal{S}$ with the plane through the center of $\mathcal{S}$ that is perpendicular to the line through p and the center of $\mathcal{S}$ (See Figure 2a). For a line l in the plane, $D(l)$ is the pair of antipodal points that are the intersection points of $\mathcal{S}$ and the line through the center of S that is perpendicular to the plane through l and the center of $\mathcal{S}$ (See Figure 2b). This duality preserves incidence relations in the sense that a point p that is incident to a line l in the plane is mapped to a great circle $D(p)$ on $\mathcal{S}$ that is incident to the two points of $D(l)$.

Recall that an *ordinary* intersection point in an arrangement of curves is an intersection point through which precisely two curves pass.



(a) The dual $D(p)$ of a point p is the great circle that is the intersection of $\mathcal{S}$ with the plane through the center of $\mathcal{S}$ that is perpendicular to the line through p and the center of $\mathcal{S}$.

(b) The dual $D(l)$ of a line l is the pair of antipodal points that are the intersection points of $\mathcal{S}$ and the line through the center of S that is perpendicular to the plane through l and the center of $\mathcal{S}$.

Figure 2: Duality between the plane and the unit sphere

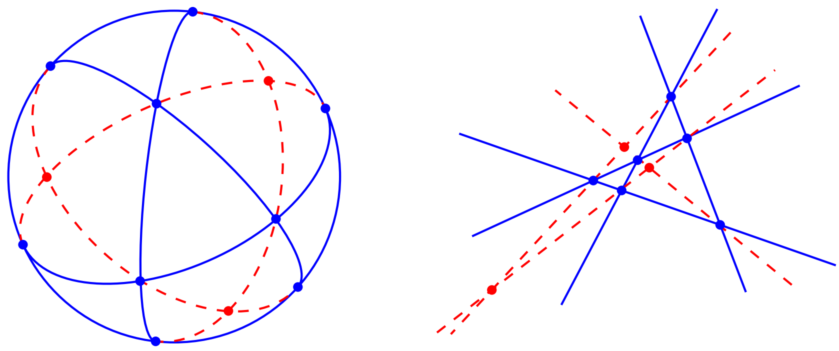


Figure 3: A set of great circles in the exceptional configuration of Theorem 3. Only the front half of the sphere is shown. One blue circle forms the boundary of the visible half-sphere and is shown as a whole. The right part shows a corresponding arrangement of lines. The dashed circles/lines form the set A. Their weight is $\frac{1}{2}$. The solid circles/lines form the set B, and their weight is $\frac{1}{4}$.

Theorem 3. *Let A and B be two nonempty sets of distinct great circles on a sphere $\mathcal{S}$. Assume that all the ordinary intersection points determined by $A \cup B$ are precisely all the intersection points determined by A. Assume further that every circle in $A \cup B$ is assigned a positive weight such that the sum of the weights of all circles incident to any given intersection point on $\mathcal{S}$ is 1. Then the configuration of circles $A \cup B$ is the sphere-dual of a failed Fano configuration that is equal, up to a projective transformation, to the one shown in Figure 1. The set A consists of the circles dual to the points whose weight is $\frac{1}{2}$.*

Figure 3 shows the exceptional configuration of Theorem 3, together with a central projection through the center of $\mathcal{S}$ on a plane that touches $\mathcal{S}$. Under this projection every two antipodal points on $\mathcal{S}$ are projected to the same point in the plane.

2 Proof of Theorem 3

We refer to the circles in A as *red* circles and to the circles in B as *blue* circles. In all remaining figures in this paper the solid lines represent the blue circles, while the dashed lines represent the red circles.

First of all, we can assume that $|A| \geq 3$ in Theorem 2, and hence, by duality, also in Theorem 3: any noncollinear set of points determines at least 3 ordinary lines (see [KM58]), and this would be

object of $\mathcal{B}$	$ch(\cdot)$	$ch_1(\cdot)$	$ch_2(\cdot)$	$ch_3(\cdot)$	$ch_4(\cdot)$
bad crossing point	-1	0	0	0	0
good crossing point	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0
bad (but not evil) triangle	0	0	$-1/4$	≥ 0	≥ 0
evil triangle	0	0	$-1/4$	$-1/4$	0
0-quadrangle	1	1	≥ 0	≥ 0	≥ 0
1-quadrangle	1	≥ 0	≥ 0	≥ 0	≥ 0
good 2-quadrangle	1	≥ 0	≥ 0	≥ 0	≥ 0
bad 2-quadrangle	1	-1	0	0	0
0-pentagon	2	2	$\geq 3/4$	≥ 0	≥ 0
1-pentagon	2	≥ 1	$\geq 3/4$	$\geq 1/2$	$\geq 1/2$
2-pentagon	2	≥ 0	≥ 0	≥ 0	≥ 0
r-(k-gon) , $k \geq 6$, $r \leq \lfloor \frac{k}{2} \rfloor$	$k-3$	$\geq k-3-r$	$\geq \frac{3}{4}k-3-\frac{r}{2}$	$\geq \frac{1}{2}k-3$	$\geq \frac{1}{2}k-3 \geq 0$

Table 1: Charge of objects of $\mathcal{B}$ before and after Steps 1–4

impossible if $|A| \leq 2$.

We consider the arrangement $\mathcal{B}$ of the circles in B on the sphere $\mathcal{S}$. Observe that any crossing point on a circle $b \in B$, even with a circle in A, is a crossing point in $\mathcal{B}$. Indeed, otherwise either it is an ordinary intersection point on b, or it is an intersection point that is not ordinary of at least two circles in A, contrary to our assumptions. It follows that $|B| \geq 2$, since otherwise there would be no crossing points in $\mathcal{B}$.

We will use the term 'triangle' for a face of $\mathcal{B}$ with three edges, the term 'quadrangle' for a face with four edges, etc. There are no faces with only two edges in $\mathcal{B}$. Indeed, otherwise all blue circles pass through the same two antipodal points p and p' on the sphere $\mathcal{S}$. As $|A| \geq 3$, there is a circle in A not passing through p, and hence also not through its antipodal point p'. This circle intersects the circles in B in ordinary intersection points, a contradiction.

Two faces in $\mathcal{B}$ are called *adjacent* if they share an edge. Similarly, two edges in $\mathcal{B}$ are called adjacent, if they are incident to the same crossing point. A great circle $s \in B$ and a face f of $\mathcal{B}$ are called adjacent, if s includes an edge of f. We begin by assigning a *charge* $ch(\cdot)$ to the faces and vertices of the arrangement $\mathcal{B}$: The charge of a face with k edges is $k-3$, while the charge of a crossing point of k blue circles is $k-3$. For every $k \geq 3$ denote by f_k the number of faces in $\mathcal{B}$ with k edges, and by t_k the number of crossing points of exactly k blue circles. It follows from Euler's formula that $\sum_k (k-3)f_k + \sum_k (k-3)t_k + 6 = 0$. Therefore, the overall charge is -6.

Our plan is to redistribute the charges (*discharge*) in four steps, such that finally every face and crossing point in $\mathcal{B}$ will have a nonnegative charge. Then it will follow that the total charge is nonnegative, hence a contradiction. For each $i = 1, 2, 3, 4$ we will denote by $ch_i(\cdot)$ the charge of an object (a face in $\mathcal{B}$ or a crossing point of blue circles) after the i-th step. For convenience, Table 1 summarizes the charges of selected objects from $\mathcal{B}$ through the four steps of discharging.

The only elements whose initial charge is negative are crossing points through which there are precisely two blue circles. We call such a crossing point *bad*.

The following lemma and the subsequent claim will be useful throughout the analysis of the discharging steps.

Lemma 1. *Let A and B as in Theorem 3. Assume that there is a blue circle s with only three pairs of antipodal vertices. If two of these points (and their antipodals) have precisely two blue circles passing through them, then $A \cup B$ is the sphere-dual of a failed Fano configuration.*

Proof. Let x_1, x_2, x_3 and their antipodal vertices x'_1, x'_2, x'_3 be the vertices on s. Since every red circle must intersect s in a distinct pair of vertices, it follows that $|A| = 3$, and there are three red circles r_1, r_2, r_3 passing through y_1, y_2, y_3, respectively, see Figure 4.

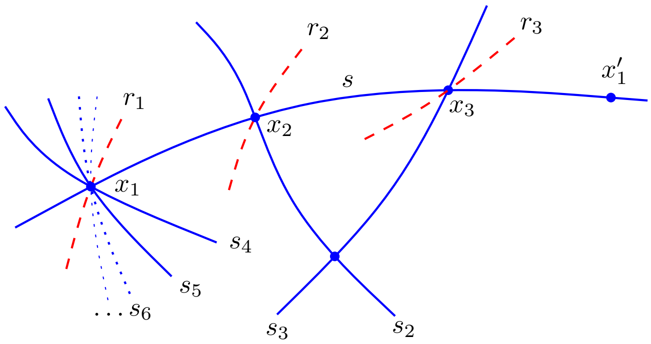


Figure 4: A blue circle with only three vertices leads to the dual of a failed Fano configuration.

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Suppose x_2 and x_3 are vertices that are incident to precisely two blue circles each, and let s_2 and s_3 denote the blue circles (different from s) passing through x_2 and x_3,

We claim that there are at most four blue circles in B. Indeed, all additional blue circles $s_4, \ldots, s_k$ must cross s at x_1 (and x'_1). The circles $r_1, s, s_4, s_5, \ldots, s_k$ all go through x_1, and hence they cross s_2 at distinct points. There are at most three points where they can cross s_2 at intersection points that are not ordinary: x_2; the intersection point of s_2 and s_3; and the intersection point of s_2 and r_3. If more than three circles go through x_1, at least one of them would have to cross s_2 at an ordinary intersection point. Since there are no ordinary intersection points on blue circles, it follows that $|B| = k \leq 4$. On the other hand, since at least two blue circles must pass through x_1, we know that there is at least a fourth blue circle s_4, and hence $|B| = 4$.

If s_4 would go through the intersection of s_2 and s_3, this would mean that s_4 contains exactly four crossing points in $\mathcal{B}$. Since the circles in A cross s_4 only at vertices of $\mathcal{B}$, and $|A| = 3$, there must be two circles of A crossing s_4 at the same point, which is impossible.

It follows that no three of the four circles of B are concurrent. There is only one combinatorial type of arrangement $\mathcal{B}$ with this property, which is shown in Figure 3. Now it can be directly seen by inspection that there is only one way to extend this arrangement $\mathcal{B}$ of four blue circles by three red circles such that the red-blue crossings occur only at the vertices of $\mathcal{B}$, and at most one red circle passes through each vertex of $\mathcal{B}$. Hence, $A \cup B$ must be the sphere-dual of a failed Fano configuration. □

We can now easily derive the following claim.

Claim 1. *Assume that there is a quadrangle d in $\mathcal{B}$ such that there are precisely two blue circles through every vertex of d, and d is adjacent to two triangles at two of its opposite edges, see Figure 5. Then $A \cup B$ is the sphere-dual of a failed Fano configuration.*

Proof. Let t_1 and t_2 denote the two triangles adjacent to d at two of its opposite edges. Let s_1, s_2, s_3. and s_4 denote the four blue circles that include the edges of d in counterclockwise order so that s_1 and s_3 separate d from t_1 and t_2, respectively. Let x_1, x_2, x_3, and x_4 denote the four vertices of d listed in counterclockwise order so that x_1 is the intersection point of s_1 and s_2. Since s_1 and s_2 are the only blue circles through x_1 and s_4 and s_1 are the only blue circles through x_4, it follows that s_2 and s_4 meet at a vertex of t_1 that we denote by x_5. Similarly, s_2 and s_4 meet at a vertex of t_2 that we denote by x'_5. x_5 and x'_5 are therefore two antipodal points on the sphere $\mathcal{S}$. Therefore, x_1, x_2, x_5 and their antipodal points on $\mathcal{S}$ are the only intersection points on s_2. Since there are precisely two blue circles through x_1 and through x_2, Lemma 1 applies, and the proof is complete. □

We proceed by describing the four discharging steps and analyzing their effect on the charges of the faces and intersection points of $\mathcal{B}$.

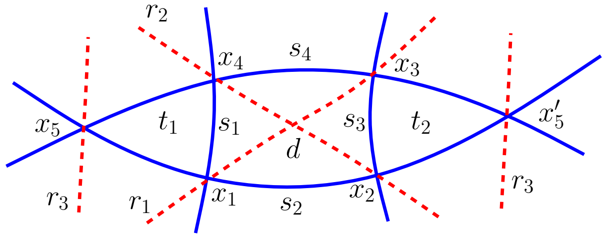
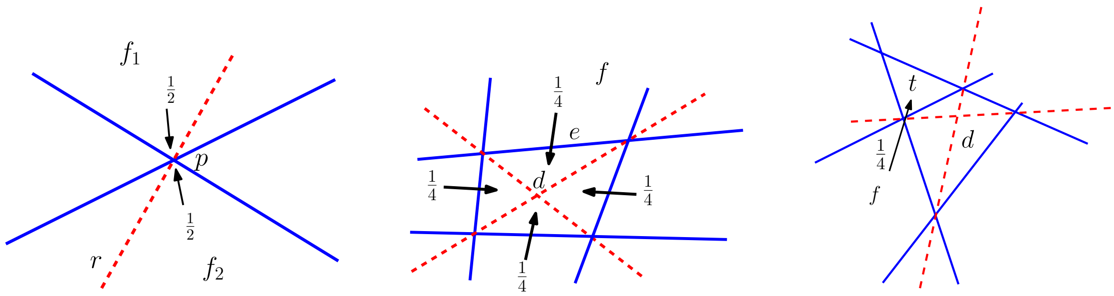


Figure 5: d is adjacent to two triangles at two of its opposite edges



(a) Step 1 (b) Step 2 (c) Step 3

Figure 6: The discharging steps 1–3

Step 1 [charging bad crossing points]: Let $\mathcal{C}$ denote the arrangement of all circles in $A \cup B$. Since we assume that no ordinary intersection point in $\mathcal{C}$ lies on a blue circle and that every pair of red circles cross at an ordinary point in $\mathcal{C}$, it follows that through each bad crossing point p in $\mathcal{B}$ there is precisely one red circle r. Let f_1 and f_2 be the two faces in $\mathcal{B}$ that are incident to p and are crossed by r (see Figure 6a). Then, we transfer $1/2$ units of charge from each of f_1 and f_2 to p.

After Step 1 every crossing point of blue circles has a nonnegative charge. Let us now examine the remaining charge at the faces of the arrangement $\mathcal{B}$. A red circle can cross the boundary of a face in $\mathcal{B}$ only at its vertices, for otherwise we would have either an ordinary intersection point of $\mathcal{C}$ on a blue circle, or an intersection point of two (or more) red circles that is not ordinary in $\mathcal{C}$. Thus, every red circle that crosses a face f in $\mathcal{B}$ induces, in fact, a red diagonal in f. A face f with m such red diagonals loses at most m units of charge in Step 1. We use an integer before the name of a face in $\mathcal{B}$ to denote the number of its red diagonals. For example, a 2-hexagon is a face with six edges in $\mathcal{B}$ that has two red diagonals. Since triangles cannot have a (red) diagonal, we refer to them simply as 'triangles' instead of 0-triangles. Thus, triangles do not lose charge in Step 1. More generally, a k-gon can have at most $\lfloor \frac{k}{2} \rfloor$ red diagonals. Pentagons may have at most two red diagonals, and thus they remain with a nonnegative charge as well. The only elements whose charge might be negative after Step 1 are 2-quadrangles, as their charge might be -1, in case they are incident to four bad crossing points.

A crossing point x of circles from $\mathcal{B}$ is called *good*, if there is a (necessarily one) red circle through x and at least 3 blue circles through x. We call a 2-quadrangle *good*, if it is incident to a good crossing point. We call a 2-quadrangle that is not incident to any good crossing point a *bad* 2-quadrangle.

Claim 2. *Any good 2-quadrangle is incident to at least two good crossing points.*

Proof. Assume to the contrary that d is a good 2-quadrangle that is incident to precisely one good crossing point x. Let s_1, s_2, s_3, and s_4 denote the four circles in $\mathcal{B}$ that constitute the edges of d in counterclockwise order so that s_1 and s_4 are incident to x. As x is a good crossing point, there is another blue circle through x that we denote by s_0. (See Figure 7.) By our assumption, all the crossing points that are incident to d, with the exception of x, are incident to precisely two blue circles and one

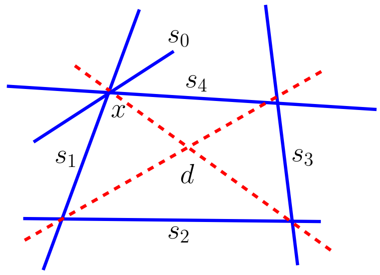


Figure 7: A good 2-quadrangle cannot be incident to exactly one good crossing point.

red circle.

Let $W(s)$ denote the weight assigned to a circle s. As we observed, for every $s \in A$ we have $W(s) = 1/2$, and for every $s \in B$, $0 < W(s) < 1/2$. Considering the crossing point of s_1 and s_2, we see that $W(s_1) + W(s_2) = 1/2$. Similarly, considering the crossing point of s_2 and s_3, we see that $W(s_2) + W(s_3) = 1/2$, and therefore $W(s_1) = W(s_3)$. Considering the crossing point of s_3 and s_4, we see that $W(s_3) + W(s_4) = 1/2$. Therefore, $W(s_1) + W(s_4) = W(s_3) + W(s_4) = 1/2$. But this is a contradiction because considering the circles through x we see that $W(s_1)+W(s_4) \leq 1/2-W(s_0) < 1/2$. (This is actually the only place in the proof of Theorem 3 where the assumption on the weights is used.) □

As a corollary of Claim 2, we conclude that after Step 1 every good 2-quadrangle has a nonnegative charge, as it is incident to at most two bad crossing points. We still have to take care of the bad 2-quadrangles. This will be done in the next step.

Step 2 [charging bad 2-quadrangles]: In this step every bad 2-quadrangle compensates for its charge shortage by taking 1/4 units of charge from each of its four neighboring faces. That is, let f be a face in $\mathcal{B}$ adjacent to a bad 2-quadrangle d, then d takes the 1/4 units of charge from the charge of f (see Figure 6b). Note that in such a case f does not have red diagonals at the vertices of the edge common to f and d.

It is easy to check, by considering the different possibilities for f, that the only elements that might have a negative charge after Step 2 are triangles adjacent to bad 2-quadrangles. (We refer the reader to the proof of Claim 4 below for a detailed argument for the case of k-gons with $k \geq 5$.) We call a triangle that is adjacent to a bad 2-quadrangle a *bad* triangle.

Claim 3. *A triangle can share an edge with at most one bad* 2*-quadrangle, unless $A \cup B$ is the sphere-dual of a failed Fano configuration.*

Proof. Let t be a triangle adjacent to two bad 2-quadrangles d_1 and d_2. Let s_1, s_2, and s_3 denote the three blue circles that constitute the triangle t, such that s_1 and s_2 separate t from d_1 and d_2, respectively (see Figure 8). The red circle r through the intersection point of s_1 and s_2 crosses s_3 at a vertex x_1 of d_1 and at a vertex x_2 of d_2, which are therefore antipodal points on the sphere $\mathcal{S}$. Since there are precisely two blue circles through each vertex of t on s_3, we can apply Lemma 1 to s_3 and conclude that $A \cup B$ is the sphere-dual of a failed Fano configuration. □

It follows that the only negative charges are now the charges of $-1/4$ at the bad triangles.

Step 3 [charging some of the bad triangles]: In this step we use the excess charge that exists at faces with at least five edges to charge part of the bad triangles.

Let f be a face in $\mathcal{B}$ with k edges, where $k \geq 5$. Let t be a bad triangle adjacent to a bad 2-quadrangle d. We transfer 1/4 units of charge from f to t, if f and t share a vertex and f shares an edge with d (see Figure 6c).

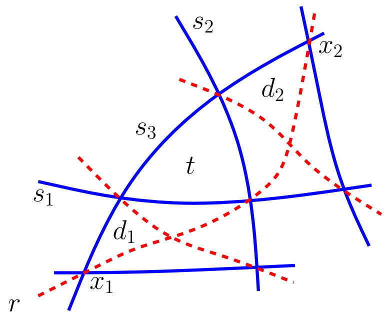


Figure 8: A bad triangle cannot be adjacent to two bad quadrangles.

Before continuing to the last step, we show that after Step 3, every face f with at least five edges remains with a nonnegative charge.

Claim 4. *Let f be a face with k edges, where $k \geq 5$. Then after Step 3, f has a nonnegative charge.*

Proof. Let r be the number of red diagonals of f. Assume first that $k \geq 6$. After Step 1, the charge of f is at least $k-3-r$. f has exactly $k-2r$ vertices that are not incident to a red diagonal, and hence at most $k-2r$ edges none of whose vertices is incident to a red diagonal of f. It follows that f may be adjacent to at most $k-2r$ (bad) 2-quadrangles. Therefore, the charge of f after Step 2 is at least $k-3-r-\frac{k-2r}{4}$. As f may contribute 1/4 units of charge to at most $k-2r$ bad triangles, the charge of f after Step 3 is at least $k-3-r-\frac{k-2r}{2}=\frac{k}{2}-3 \geq 0$.

It remains to consider the case where f is a pentagon. If f is a 2-pentagon, then f cannot be adjacent to any (bad) 2-quadrangle. Therefore, Step 2 as well as Step 3 do not affect the charge of f and it remains at least 0, as it is after Step 1. If f is a 1-pentagon, then after Step 1 the charge of f is at least 1. f may be adjacent to at most one 2-quadrangle. Therefore, after Step 2 the charge of f is at least 3/4. f contributes 1/4 units of charge in Step 3 to at most one bad triangle and hence remains with a charge of at least 1/2 after Step 3.

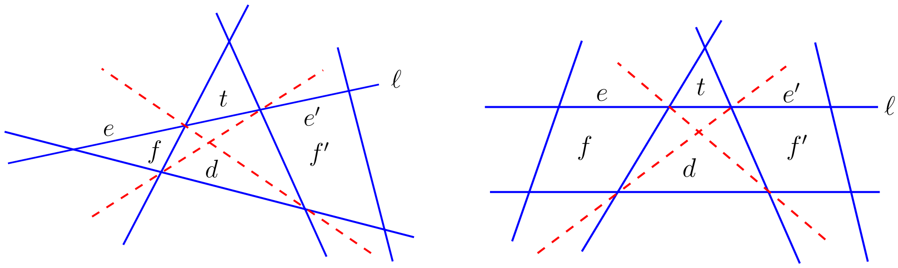
Finally, if f is a 0-pentagon, then after Step 1 the charge of f is 2. Observe that if f shares two adjacent edges e_1 and e_2 with bad 2-quadrangles d_1 and d_2, respectively, then the common vertex of e_1 and e_2 cannot be a vertex of a bad triangle t. Indeed, otherwise t is adjacent to two bad 2-quadrangles, which contradicts Claim 3, unless $A \cup B$ is the sphere-dual of a failed Fano configuration. From this observation it follows that if f is adjacent to five bad 2-quadrangles, then it does not share a vertex with any bad triangle and hence the charge of f after Step 3 is 3/4. If f shares a vertex with five bad triangles, then it may be adjacent to at most two bad 2-quadrangles (in fact one could show that even that is not possible) and hence the charge of f after Step 3 is at least 1/4. In all other cases f is adjacent to at most four bad 2-quadrangles and shares a vertex with at most four bad triangles and hence the charge of f after Step 3 is at least 0. (This last argument is far from being tight, yet it suffices for our needs.) ☐

Therefore, after Step 3, the only objects with a negative charge are those bad triangles who did not receive 1/4 units of charge in Step 3. We call those triangles *evil*.

Step 4 [charging evil triangles]: After Step 3 of discharging, the only elements without the desired charge are evil triangles, as they have $-1/4$ units of charge each. We will use the excess charge that exists at the 0-quadrangles to move 1/4 units of charge to every evil triangle.

For every 0-quadrangle q, consider the set E of edges of q that are not edges of bad 2-quadrangles. Then the charge of q after Step 3 is $|E|/4$. For every $e \in E$ let $\ell_e \in B$ be the great circle that includes e. We call the pair (ℓ_e, q) a *helping pair* and we designate 1/4 unit from the charge of q to the pair (ℓ_e, q).

For any evil triangle t, let d be the bad 2-quadrangle adjacent to it, and let $\ell \in B$ be the great



(a) An evil pair (ℓ, d) of degree 1 in H (b) An evil pair (ℓ, d) of degree 2 in H

Figure 9: Evil pairs

circle that separates t and d. We call the pair (ℓ, d) an *evil pair*. We will show that there are at least as many helping pairs as there are evil pairs. Thus we will successfully charge each evil triangle with $1/4$ units of charge taken of the excess charge at the 0-quadrangles after Step 3.

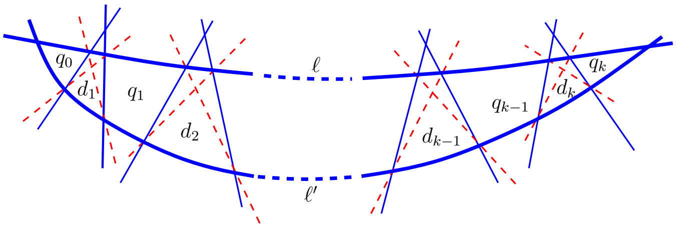
Define a bipartite graph H whose vertices are the evil pairs and the helping pairs. Let (ℓ, d) be an evil pair, let t be the evil triangle adjacent to d and ℓ, and let f and f' be the two faces in $\mathcal{B}$, other than t, that are adjacent to both ℓ and d (see Figure 9). Let e and e' be the edges of f and f', respectively, on ℓ. Since t is evil, f and f' can be either triangles or 0-quadrangles. Moreover, the edges e and e' cannot be edges of bad 2-quadrangles, as d is the only bad 2-quadrangle adjacent to t. Each of (ℓ, f) and (ℓ, f') is a helping pair, assuming f or f', respectively, are not triangles. If f is not a triangle, we connect (ℓ, d) in H to the helping pair (ℓ, f). Similarly, if f' is not a triangle, we connect (ℓ, d) in H to the helping pair (ℓ, f'). Observe that if both f and f' are triangles, then by Claim 1, $A \cup B$ is the sphere-dual of a failed Fano configuration. Therefore, we may assume that the degree in H of every evil pair is either 1 or 2. The degree in H of every helping pair is at most 2, because a helping pair (ℓ, q) may be connected only to evil pairs (ℓ', d) such that $\ell = \ell'$ and d is adjacent to q. It follows that the connected components of H that include evil pairs are either paths alternating between evil pairs and helping pairs, or theoretically, even cycles alternating between evil pairs and helping pairs. Therefore, in order to show that there are at least as many helping pairs as there are evil pairs, it is enough to show that no connected component in H is a path both of whose end vertices are evil pairs.

Indeed, assume to the contrary that there is such a connected component in H. Let its vertices be $(\ell, d_1), (\ell, q_1), \ldots, (\ell, q_{k-1}), (\ell, d_k)$, so that for every $1 \leq i \leq k-1$, (ℓ, q_i) is a helping pair connected to both (ℓ, d_i) and (ℓ, d_{i+1}). It follows that there is a great circle $\ell' \in B$ that includes all edges of $d_1, \ldots, d_k$ and $q_1, \ldots, q_{k-1}$ that are opposite to those included in ℓ.

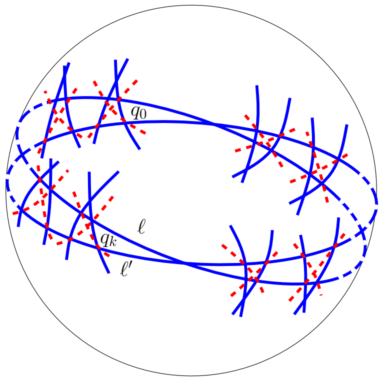
Since the degree in H of (ℓ, d_1) is 1, the face in $\mathcal{B}$, other than q_1, adjacent to both ℓ, ℓ', and to d_1 must be a triangle which we denote by q_0. Similarly, the face in $\mathcal{B}$, other than q_{k-1}, adjacent to both ℓ, ℓ', and to d_k must be a triangle which we denote by q_k. Observe that ℓ and ℓ' meet at a vertex of q_0 and at a vertex of q_k (see Figure 10).

We claim that the only triangles in $\mathcal{B}$ adjacent to ℓ' are q_0, q_k, and of course their antipodal triangles on the sphere $\mathcal{S}$. This is because for every $0 \leq i \leq k$, the face adjacent to ℓ' that shares an edge with q_i cannot be a triangle as it admits a red diagonal at least at one of its vertices. And moreover, we may assume that for every $1 \leq i \leq k$, the face adjacent to ℓ' that shares an edge with d_i is not a triangle. Indeed, otherwise by Claim 1, $A \cup B$ is the sphere-dual of a failed Fano configuration (recall that there is an evil triangle adjacent to d_i on the other side of ℓ). This is a contradiction to a theorem of Levi [Le26] saying that in any nontrivial arrangement of lines in the real projective plane, every line must be adjacent to at least 3 triangular faces. (Here, we apply Levi's theorem after identifying antipodal points on the sphere $\mathcal{S}$ and thus reducing the great circles in $A \cup B$ to a set of lines in the projective plane.) See [Fe04, § 5.4] and [Gr72] for very short proofs of Levi's theorem.

We conclude that after Step 4, all the faces in the arrangement $\mathcal{B}$ have a nonnegative charge, and the same holds for every crossing point in $\mathcal{B}$. Thus, the overall charge is nonnegative, contradicting



(a)



(b)

Figure 10: A connected component in H both of whose endpoints are evil pairs

the fact the the total charge in the beginning was -6.

3 Notes and Concluding Remarks

Additional general position assumptions. If the arrangement $\mathcal{B}$ is in general position in the sense that no three blue circles from B pass through the same point, then Theorem 3 and hence also its dual, Theorem 2, can be strengthened as follows, leaving the proof almost as is. This theorem does not mention any weights, and it omits the requirement that the lines determined by A are ordinary.

Theorem 4. *Let A and B be two nonempty disjoint sets of points in the plane such that B is in general position. Assume that no line determined by A and no ordinary line in $A \cup B$ passes through a point of B. Then $A \cup B$ is, up to a projective transformation, the configuration in Figure 1.*

Let us see why the proof of Theorem 3 establishes Theorem 4, and the added assumptions of Theorem 3 are not needed: the point weights played their first role in defining the sets A and B and thereby transforming Theorem 1 into Theorem 2 (and its equivalent dual formulation of Theorem 3). Besides this, the only place where the weights are used is the proof of Claim 2. However, if the circles in B are in general position, then there are no good crossing points in $\mathcal{B}$, and Claim 2 is vacuously true.

In Theorem 4 we allow more than two points of A to be collinear as long as they are not collinear with a point of B. Indeed, in the proof of Theorem 3 we never use the assumption that every intersection point determined by A is an ordinary intersection point with respect to $A \cup B$, but only that no intersection point determined by circles from A lies on a circle from B.

Proving Theorem 4 for the case where B is not required to be in general position would imply the following conjecture[1]. Recall that an *ordinary point* in a point configuration P is a point $x \in P$ through which there is an ordinary line.

Conjecture 1. *Let $G = (V, E)$ be the* Sylvester graph *of a finite set of points P. That is, $V = \{\, p \in P \mid p$ is an ordinary point in $P \,\}$ and $E = \{\, (p_1, p_2) \mid p_1$ and p_2 determine an ordinary line in $P \,\}$. Then G is a complete (nonempty) graph if and only if no three points in P are collinear, or P is a failed Fano configuration.*

Geometrically induced matchings. We would like to note a corollary of Theorem 4. It is well known that the set of edges of a complete graph on $2n$ vertices can be partitioned into (necessarily

[1] This conjecture is attributed to Sylvester by Smyth [Sm89, Conjecture C].

$2n-1$) edge-disjoint perfect matchings. A nice way to realize such a partitioning is to think about the vertices of K_{2n} as the vertices of a regular $(2n-1)$-gon plus its center. Then each of the $2n-1$ directions of the edges of the $(2n-1)$-gon induces a perfect matching in which two points are matched if the straight line they determine is parallel to the direction we choose, plus matching the center with the remaining point. These $2n-1$ perfect matchings are edge-disjoint.

Now let G be a complete geometric graph on $2n$ vertices in general position in the plane. We call a matching in G *geometrically induced*, if the lines containing the edges of the matching go through a common point, which is then called the *center* of the matching. The question is, whether we can partition the set of edges of a complete geometric graph G on $2n$ vertices in general position in the plane into edge-disjoint geometrically induced perfect matchings. By Theorem 4, this is impossible unless $n = 1$ or $n = 2$. Indeed, assume it is possible and let B be the set of $2n$ vertices of G, and let A be the set of all centers of the geometrically induced perfect matchings. Then A and B satisfy the assumptions in Theorem 4.

It is an interesting open question to determine the maximum possible number of edge-disjoint geometrically induced perfect matchings of a complete geometric graph on $2n$ vertices in general position in the plane. It seems natural to conjecture that the answer should be $n+1$. This number is attained for the set of vertices of a regular $2n$-gon in the plane when n is even. Here observe that the geometrically induced perfect matchings whose centers are the points at infinity that correspond to the n directions of the edges of the regular $2n$-gon plus the center of the $2n$-gon, are all pairwise edge-disjoint.

Weakly magic configurations. One can weaken the notion of a magic configuration and omit the restriction of the weights assigned to the points being positive. In this case there seems to be a much larger variety of magic configurations, and yet not every configuration is magic. In this context it is interesting to note that given that a configuration is magic (even in the weak sense), it is very easy to assign the right weight x_i (and in a unique way) to each point p_i, just as a function of the number k_j of lines determined by the point set that pass through each point p_j. To this end let $p_1, \ldots, p_n$ denote the points of a magic configuration P. Let $Y = \sum_{i=1}^{n} x_i$ be the total weight. There are k_i lines determined by P that pass through p_i. The sum of the weights assigned to the points of P on each of these lines is 1. It follows that $Y = k_i - x_i(k_i - 1)$. Therefore, $x_i = \frac{k_i - Y}{k_i - 1}$. From the equation $Y = \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \frac{k_j - Y}{k_j - 1}$, we can express Y as

$$Y = \frac{\sum_{j=1}^{n} \frac{k_j}{k_j - 1}}{1 + \sum_{j=1}^{n} \frac{1}{k_j - 1}}, \text{ and hence, } \quad x_i = \frac{1}{k_i - 1}\left(k_i - \frac{\sum_{j=1}^{n} \frac{k_j}{k_j - 1}}{1 + \sum_{j=1}^{n} \frac{1}{k_j - 1}}\right).$$

In particular, if $k_i = k_j$, then $x_i = x_j$. It is now clear that the weight assignment is unique, if exists.

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