On the Maximum Number of Edges in Topological Graphs with no Four Pairwise Crossing Edges

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ABSTRACT

We show that the maximum number of edges in a topological graph on n vertices and with no four pairwise crossing edges is $O(n)$.

1. INTRODUCTION

A *topological graph* is a graph drawn in the plane with its vertices as points and its edges as Jordan arcs that connect corresponding points and do not contain any other vertex as an interior point. We consider *simple* graphs, that is, graphs that do not contain loops or parallel edges. We also assume that any two arcs of a topological graph have a finite number of intersection points, that are either endpoints or crossing points. A *planar* graph is a graph that can be drawn in the plane without any pair of crossing edges. One possible generalization of the notion of planar graphs are *k-quasi-planar* graphs. A *k-quasi-planar* graph is a topological graph with no k pairwise crossing edges. We denote by $f_k(n)$ the maximum number of edges in such a graph on n vertices.

Since 2-quasi-planar graphs are planar graphs, it follows from Euler's Polyhedral Formula that $f_2(n) \leq 3n-6$. It is conjectured that for any fixed k, there is a constant C_k such that $f_k(n) \leq C_k n$. Agarwal, Aronov, Pach, Pollack, and Sharir [2] were the first to prove this conjecture for $k=3$. Later, Pach, Radoičić, and Tóth [3] simplified their proof and showed that $f_3(n) \leq 65n$. Recently, Ackerman and Tardos [1] proved that $7n - O(1) \leq f_3(n) \leq 8n - \Omega(1)$, and provided a tight bound of $6.5n - \Omega(1)$ for *simple* topological graphs (graphs in which every pair of edges intersect at most once). For a fixed $k \geq 4$, the best upper bound for topological graphs is $O(n \log^{4k-12} n)$ [3], while for topological graphs with x-monotone edges, Valtr [4] showed an upper bound of $O(n \log n)$. In the following section we prove that $f_4(n) = O(n)$.

2. PROOF OF THE MAIN THEOREM

Since the (underlying abstract) graphs we consider are simple, we have $f_4(1) = 0$ and $f_4(2) = 1$. For greater values of n we prove the following theorem.

THEOREM 1. *For any integer $n > 2$, every topological graph on n vertices with no four pairwise crossing edges has at most $36(n-2)$ edges.*

PROOF. It is easy to see that $f_4(3) = 3 < 36(3-2)$. Let G be a topological graph on $n > 3$ vertices and without four pairwise crossing edges. We denote by $V(G)$ the vertex set of G, and by $E(G)$ the edge set of G. Given an edge $e \in E(G)$ and two points p and q on e, we will use the notation $e|_{p,q}$ to denote the segment of e between p and q. For a vertex v we denote by $d(v)$ the degree of v. If there is a vertex $v \in V(G)$ such that $d(v) = 1$, then we can conclude the theorem by induction. Hence, we assume the degree of every vertex in G is at least two. Assume, w.l.o.g., that G is drawn with the least possible number of crossings, such that there are no four pairwise crossings, and that there are no three edges crossing at the same point. Let e_1 and e_2 be two edges of G that intersect at least twice. A region bounded by segments of e_1 and e_2 that connect two consecutive intersection points is called a *lens*. We observe, as in [3], that G has no *empty* lenses, that is, lenses which do not contain a vertex of G. If there were empty lenses, then the number of crossings in G could be reduced. For the same reason G does not contain self-intersecting edges.

Let G' be the (drawing of the) planar graph *induced* by G. That is, $V(G') = V(G) \cup X(G)$, where $X(G)$ is the set of crossing points in G; and $e' \in E(G')$ if e' is a segment of an edge of G that connects two vertices in $V(G')$ and contains no other vertex from $V(G')$. We refer to the edges of G' as *p-edges*, in order to distinguish them from the edges of G. Denote by $F(G')$ the set of faces of G', and let $|f|$ be the number of p-edges along the boundary of a face $f \in F(G')$.[1] Given a face f, we denote by $v(f)$ the number of vertices from $V(G)$ along the boundary of f (we call these vertices *original* vertices). We will use the terms *triangles*, *quadrilaterals*, *pentagons*, and *hexagons* to refer to faces of size 3,4,5, and 6, respectively. An integer m before the name of a face, denotes the number of original vertices on its boundary. For example, a 2-pentagon is a face of size 5 that has 2 original vertices on its boundary. We proceed, as in [1], by assigning charges to the faces of G', such that each face f receives a charge of $|f| + v(f) - 4$. Summing the total charges over all the faces of G' we have

$$\sum_{f \in F(G')} \Big(|f| + v(f) - 4\Big) =$$
$$2|E(G')| + \sum_{f \in F(G')} v(f) - 4|F(G')| = 4n - 8, \quad (1)$$

where the last equality follows from Euler's formula and

[1] Note that a p-edge can sometimes appear twice along the boundary of a face.

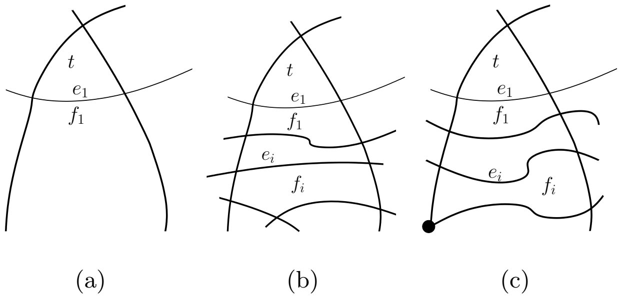


Figure 1: Charging a 0-triangle

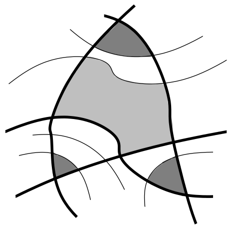


Figure 2: A pentagon contributing to three 0-triangles through non-consecutive p-edges implies four pairwise crossing edges.

from the next equalities:

$$\sum_{f\in F(G')} v(f) = \sum_{u\in V(G)} d(u) = \sum_{u\in V(G')} d(u) - \sum_{u\in X(G)} d(u) = 2|E(G')| - 4\Big(|V(G')| - |V(G)|\Big).$$

A *wedge* is a triplet $w = (v, l, r)$, such that $v \in V(G)$, l and r are edges emerging from v, and l immediately follows r in a clockwise order of the edges touching v. Our plan is to re-distribute the charges, such that there will be no faces with a negative charge and every wedge will be charged with at least $\frac{1}{18}$ units of charge. Then, the total charge over all the wedges will be $\frac{2|E(G)|}{18} \le 4n - 8$, and the theorem will follow. (Note that the number of wedges in which a vertex $v \in V(G)$ participates is $d(v)$. Here we use the assumption that the degree of every vertex is at least two.) Since a face of size one yields a self-intersecting edge, and a face of size two yields an empty lens or two parallel edges, the only faces with a negative charge are 0-triangles. We proceed by describing a method to charge these faces. Then, we will show how to charge the wedges of original vertices.

Charging 0-triangles. Let t be a 0-triangle, let e_1 be one of its p-edges, and let f_1 be the other face incident to e_1 (see Figure 1(a)). It must be that $|f_1| > 3$, for otherwise there would be an empty lens. If $v(f_1) > 0$ or $|f_1| > 4$, we move $\frac{1}{3}$ units of charge from f_1 to t, and say that f_1 *contributed* $\frac{1}{3}$ units of charge to t *through* e_1. Otherwise, f_1 must be a 0-quadrilateral. Let e_2 be the opposite p-edge to e_1 in f_1, and let f_2 be the other face incident to e_2. Applying the same arguments as above, we conclude that either f_2 contributes $\frac{1}{3}$ units of charge to t through e_2, or f is also a 0-quadrilateral. In the second case we continue to the next face, that is, the other face that is incident to the opposite p-edge to e_2 in f_2. However, at some point we must encounter a face that is not a 0-quadrilateral. Denote by f_i this face, by f_{i-1} the face preceding f_i, and by e_i the edge incident to both of these faces. Then f_i will contribute $\frac{1}{3}$ units of charge to t through e_i (see Figures 1(b,c)).

In a similar way t obtains $\frac{2}{3}$ units of charge from its other p-edges. Thus, after re-distributing charges this way, the charge of every 0-triangle is 0. Note that a face can contribute through each of its p-edges at most once. Therefore, every face f such that $|f| + v(f) \ge 6$ still has a non-negative charge. It remains to verify that 1-quadrilaterals and 0-pentagons, which had only one unit of charge to contribute, also have a non-negative charge. Indeed, a 1-quadrilateral contributes to at most two 0-triangles, since the endpoints of a p-edge through which it contributes must be vertices from $X(G)$. A 0-pentagon, on the other hand, can contribute to at most three 0-triangles by the following easy observation.

OBSERVATION 2.1. *A* 0*-pentagon contributes charge to at most three* 0*-triangles. Moreover, if it contributes to three* 0*-triangles, then the contribution must be done through consecutive p-edges.*

PROOF. One can easily inspect that a contribution to three 0-triangles through non-consecutive p-edges implies four pairwise crossing edges (see Figure 2). In case a 0-pentagon contributes to more than three 0-triangles, then there must be three non-consecutive p-edges through which it contributes. $\square$

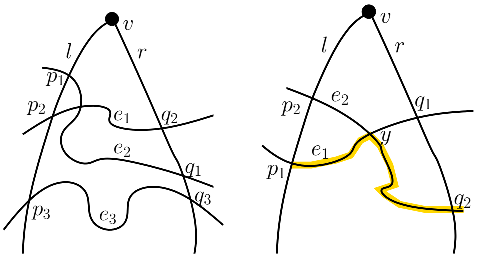
Charging wedges. After the previous step, the faces with a zero charge, apart from 0-triangles and 0-quadrilateral, are 0-pentagons that contributed to three 0-triangles, and 0-hexagons that contributed to six 0-triangles. We call such faces *bad* faces. Faces that have a positive charge are called *good* faces. Our goal now, is to find some extra charge for each wedge. This extra charge will be found next to a *farthest uncut $\mathcal{A}$-crossing* or *$\mathcal{X}$-crossing* of the wedge. We begin with a few definitions.

Let $w = (v, l, r)$ be a wedge, and let e be an edge crossing l at p and r at q, such that $e|_{p,q}$ does not cross l or r. We denote by $w|_{e,p,q}$ the area to the left of the closed curve formed by traversing from v on $l|_{v,p}$, $e|_{p,q}$, and $r|_{q,v}$.

DEFINITION 2.2 (UNCUT $\mathcal{A}$-CROSSING). *Let $w = (v, l, r)$ be a wedge. An $\mathcal{A}$*-crossing *of w is a triplet $cr = (e, p, q)$ such that e is an edge crossing l at p and r at q such that: (1) $e|_{p,q}$ does not intersect l or r; and (2) the endpoints of e are not in $w|_{e,p,q}$. We say that cr is an* uncut *$\mathcal{A}$-crossing of w, if $e|_{p,q}$ is a p-edge. For examples, refer to Figure 3(a).*

Let $cr = (e, p, q)$ be an $\mathcal{A}$-crossing of a wedge $w = (v, l, r)$. We use the notation $w|_{cr}$ as an abbreviation for $w|_{e,p,q}$, and say that cr is an *empty* $\mathcal{A}$-crossing of w, if there are no original vertices in $w|_{cr}$. Given another $\mathcal{A}$-crossing of w, $cr' = (e', p', q')$, we say that cr is *farther* than cr' (and cr' is *closer* than cr), if $p' \in l|_{v,p}$ and $q' \in r|_{v,q}$. Clearly, not every two $\mathcal{A}$-crossings of a wedge are comparable, but uncut $\mathcal{A}$-crossings are.

DEFINITION 2.3 ($\mathcal{X}$-CROSSING). *Let w be a wedge, and let $cr_1 = (e_1, p_1, q_1)$ and $cr_2 = (e_2, p_2, q_2)$ be two $\mathcal{A}$-crossings*



(a) $\mathcal{A}$-crossings of a wedge. (e_3, p_3, q_3) is uncut and is farther than (e_1, p_1, q_1) and (e_2, p_2, q_2).

(b) an $\mathcal{X}$-crossing of a wedge. The visible part of the $\mathcal{X}$-crossing is marked.

Figure 3: Crossing patterns of a wedge

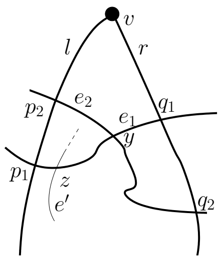


Figure 4: An illustration for the proof of Observation 2.4

*of w. Then (cr_1, cr_2) is an $\mathcal{X}$-*crossing *of w if $e_1|_{p_1,q_1}$ and $e_2|_{p_2,q_2}$ intersect exactly once.*

Let $x = \big(cr_1 = (e_1, p_1, q_1), cr_2 = (e_2, p_2, q_2)\big)$ be an $\mathcal{X}$-crossing of a wedge $w = (v, l, r)$. We say that x is *empty* if both cr_1 and cr_2 are empty $\mathcal{A}$-crossings. The notation $w|_x$ represents the area $w|_{cr_1} \cup w|_{cr_2}$. Assuming $p_2 \in l|_{v,p_1}$ (and therefore, $q_1 \in r|_{v,q_2}$), the boundary of $w|_x$ is the closed curve formed by $l|_{v,p_1}$, $e_1|_{p_2,y}$, $e_2|_{y,q_2}$, and $r_{q_2,v}$, where y is the intersection point of $e_1|_{p_1,q_1}$ and $e_2|_{p_2,q_2}$. We call the curve $e_1|_{p_1,y} \cup e_2|_{y,q_2}$ the *visible* part of $w|_x$ and denote it by $\mathrm{Vis}(x)$, when it is clear to which wedge we refer. We denote by $\mathrm{Vis}(x)_l$ and $\mathrm{Vis}(x)_r$ the two components of $\mathrm{Vis}(x)$, $e_1|_{p_1,y}$ and $e_2|_{y,q_2}$, respectively. See Figure 3(b) for an example. The next observation will be useful in the sequel.

Observation 2.4. *Suppose $x = \big(cr_1 = (e_1, p_1, q_1), cr_2 = (e_2, p_2, q_2)\big)$ is an empty $\mathcal{X}$-crossing of a wedge $w = (v, l, r)$, such that $Vis(x)_l \subset e_1$. Then an edge e' that crosses $Vis(x)_l$ (resp., $Vis(x)_r$) must cross l (resp., r) and must not cross e_2 (resp., e_1).*

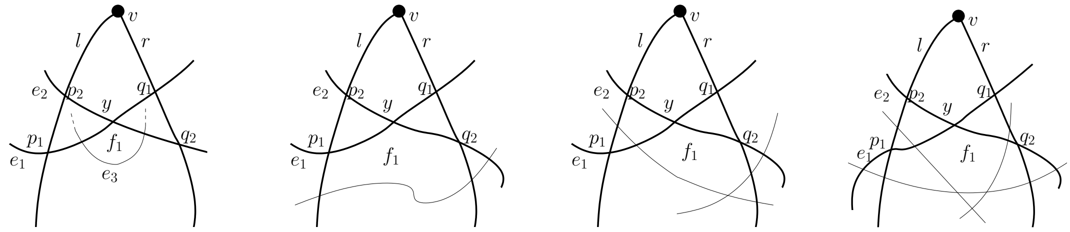
Proof. Let y be the crossing point of $e_1|_{p_1,q_1}$ and $e_2|_{p_2,q_2}$, and let z be the crossing point of e' and $e_1|_{p_1,y}$ (see Figure 4). Since x is an empty $\mathcal{X}$-crossing of w, e' must cross the boundary of $w|_x$ at least one more time. If it crosses $e_1|_{p_1,q_1}$ at another point different from z, then we have an empty lens. Therefore e' cannot cross $\mathrm{Vis}(x)$ at another point. Otherwise, if e' crosses $r|_{v,q_2}$ then it must also cross $e_2|_{p_2,y}$. This implies four pairwise crossing edges: e', e_1, e_2, and r. Thus e' must cross l. Moreover, it must not cross e_2 since this also yields four pairwise crossing edges. The proof for an edge crossing $\mathrm{Vis}(x)_r$ is similar and is thus omitted. □

Let x and x' be two $\mathcal{X}$-crossings of a wedge w. we say that x is a *farther* $\mathcal{X}$-crossing of w than x', if one $\mathcal{A}$-crossing of x is farther than one $\mathcal{A}$-crossing of x' and the other $\mathcal{A}$-crossing of x' is not farther than the other $\mathcal{A}$-crossing of x. In a similar way, we say that an uncut $\mathcal{A}$-crossing is *farther* (resp., *closer*) than an $\mathcal{X}$-crossing of the same wedge, if it is farther (resp., closer) than both $\mathcal{A}$-crossings of the $\mathcal{X}$-crossing.

Given a wedge $w = (v, l, r)$, we look for an empty uncut $\mathcal{A}$-crossing or $\mathcal{X}$-crossing of w, such that there are no empty uncut $\mathcal{A}$-crossing or $\mathcal{X}$-crossing, farther than it. If there is no such uncut $\mathcal{A}$-crossing or $\mathcal{X}$-crossing, then the face incident to v,l, and r, is not a 1-triangle. Thus, its charge is at least $\frac{1}{3}$ units, from which we use $\frac{1}{18}$ units to charge w. If there is such an empty uncut $\mathcal{A}$-crossing $cr = (e, p, q)$, then let f be the face incident to $e|_{p,q}$ outside $w|_{cr}$. f is not a triangle as this would yield an empty lens or parallel edges. Nor can it be a 0-quadrilateral since this would imply an empty uncut $\mathcal{A}$-crossing farther than cr. If f is a bad pentagon, then it follows from Observation 2.1 that there is an empty $\mathcal{X}$-crossing, farther than cr. Therefore, f must be a good face which will contribute $\frac{1}{18}$ units of charge to w through $e|_{p,q}$. It remains to consider the case in which there is an empty $\mathcal{X}$-crossing x, such that there is no empty uncut $\mathcal{A}$-crossing or $\mathcal{X}$-crossing, farther than x.

Denote by $cr_1 = (e_1, p_1, q_1)$ and $cr_2 = (e_2, p_2, q_2)$ the two $\mathcal{A}$-crossings forming x, such that $\mathrm{Vis}(x)_l = e_1|_{p_1,y}$, where y is the intersection point of $e_1|_{p_1,q_1}$ and $e_2|_{p_2,q_2}$. Let f_1 be the face that is incident to y and outside $w|_x$. Suppose f_1 is a 0-triangle and let e_3 be the third edge incident to it (see Figure 5(a)). It follows from Observation 2.4 that e_3 must cross l, thus l, e_1, e_2, and e_3 are pairwise crossing. If f_1 is a bad pentagon, then we consider the possible cases, according to whether none, one, or both of $e_1|_{p_1,y}$ and $e_2|_{y,q_2}$ are p-edges. In case both of them are p-edges (see Figure 5(b)), then there is an empty uncut $\mathcal{A}$-crossing of w that is farther than x. In case one of them, say $e_2|_{y,q_2}$, is a p-edge (see Figure 5(c)), then there is an empty $\mathcal{X}$-crossing farther than x. If none of them is a p-edge (see Figure 5(d)), then there must be four pairwise crossing edges. In a similar way, if f_1 is a bad hexagon then there must be four pairwise crossing edges, or an $\mathcal{X}$-crossing of w farther than x.

Therefore, if f_1 is not a good face, then it must be a 0-quadrilateral. Suppose f_1 is a 0-quadrilateral and let f_2 be the face outside of $w|_x$ that shares a p-edge with f_1 and is incident to $\mathrm{Vis}(x)_l$ (see Figure 5(e)). If there is no such face, or there is no face outside $w|_x$ that shares a p-edge with f_1 and is incident to $\mathrm{Vis}(x)_r$, then there is an empty $\mathcal{X}$-crossing farther than x. Examining f_2, one can see by inspection that it cannot be a 0-triangle, as this implies four pairwise crossing edges. If f_2 is a bad pentagon (see Figure 5(f,g)), or a bad hexagon, then again there must be four pairwise crossing edges or an empty $\mathcal{X}$-crossing farther than x. Thus, either f_2 is a good face or it is a 0-quadrilateral. In the second case, we examine the next face, that is, the face $f_3 \neq f_1$ such that f_3 is outside $w|_x$, shares a p-edge with f_2, and is incident to $\mathrm{Vis}(x)_l$. If there is no such a face, then we have an empty $\mathcal{X}$-crossing farther than x. Otherwise, we

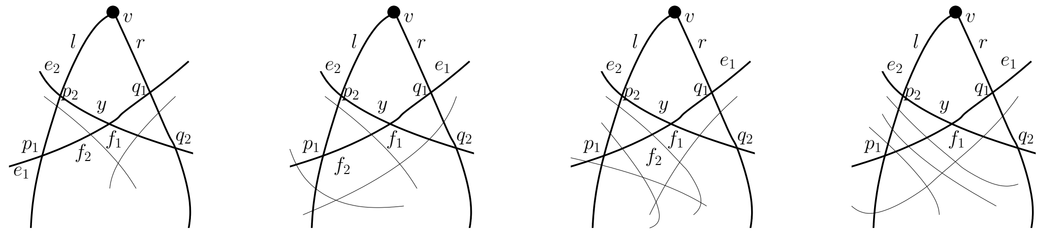


(a) f_1 is a 0-triangle

(b) f_1 is a bad pentagon, such that $e_1|_{p_1,y}$ and $e_2|_{y,q_2}$ are p-edges.

(c) f_1 is a bad pentagon, such that $e_2|_{y,q_2}$ is a p-edge and $e_1|_{p_1,y}$ is not.

(d) f_1 is a bad pentagon, such that $e_1|_{p_1,y}$ and $e_2|_{y,q_2}$ are not p-edges.



(e) f_2

(f) f_2 is a bad pentagon

(g) f_2 is a bad pentagon

(h) All the faces along $\mathrm{Vis}(x)_l$ are 0-quadrilaterals

Figure 5: Obtaining charge near a farthest $\mathcal{X}$-crossing

can apply the same arguments we used for f_2 on f_3, proceed to the next face, if f_3 is not a good face, and so on. Thus, at some point we must encounter a good face, for otherwise we have an empty $\mathcal{X}$-crossing farther than x (see Figure 5(h)). Let f_i be the first good face we encounter along $\mathrm{Vis}(x)_l$, and let e_i be the p-edge of f_i that is contained in $\mathrm{Vis}(x)_l$. Then f_i contributes $\frac{1}{18}$ units of charge to w through e_i.

Next, we prove that after charging every wedge of an original vertex, as above, there are no faces with a negative charge. For that, we need to show that a face cannot contribute to "too many" wedges. For a (good) face f and one of its p-edges m, we say that f is a *possible $\mathcal{X}$-contributor* to a wedge w through m, if there is an empty $\mathcal{X}$-crossing of w, x, such that f is outside $w|_x$ and $m \subset \mathrm{Vis}(x)_l$.

Observation 2.5. *Let f be a face and let m be one of its p-edges. Then f is a possible $\mathcal{X}$-contributor through m to at most one wedge.*

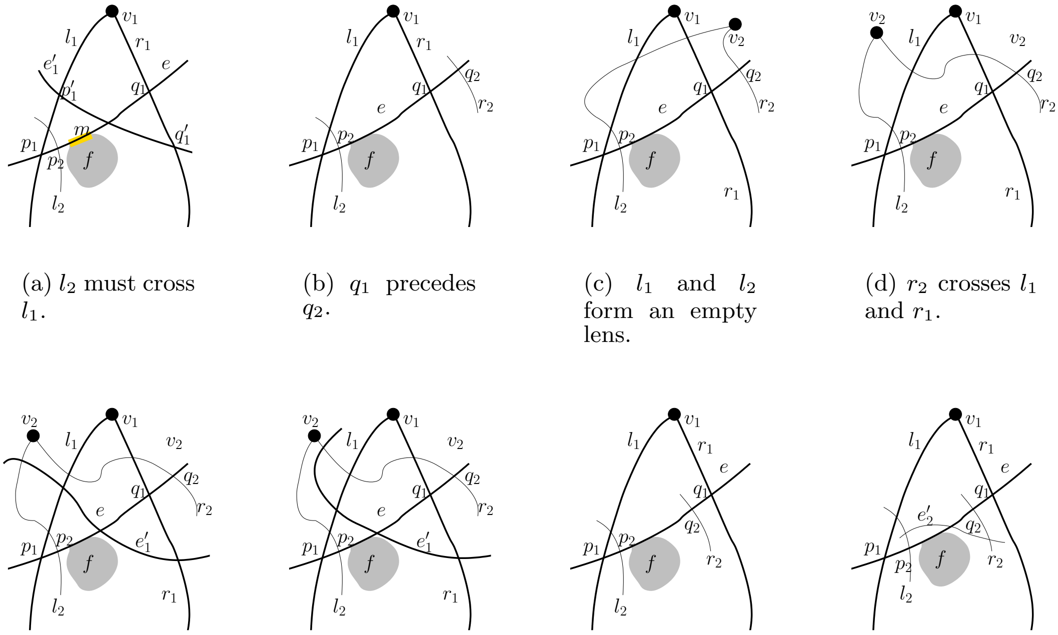
Proof. Suppose there is a face f that is a possible $\mathcal{X}$-contributor through one of its p-edges, m, to two wedges, $w_1 = (v_1, l_1, r_1)$ and $w_2 = (v_2, l_2, r_2)$. Let e be the edge containing m, then there are four points p_1, q_1, p_2, q_2, such that $cr_1 = (e, p_1, q_1)$ is an $\mathcal{A}$-crossing of w_1 and $cr_2 = (e, p_2, q_2)$ is a $\mathcal{A}$-crossing of w_2. Denote by $x_1 = \big((e, p_1, q_1), (e'_1, p'_1, q'_1)\big)$ the empty $\mathcal{X}$-crossing of w_1, such that $m \subset \mathrm{Vis}(x_1)_l$, and by $x_2 = \big((e, p_2, q_2), (e'_2, p'_2, q'_2)\big)$ the empty $\mathcal{X}$-crossing of w_2, such that $m \subset \mathrm{Vis}(x_2)_l$. Suppose we sort p_1, q_1, p_2, q_2 by the order in which they appear when traversing e from one of its endpoints to the other, such that when traversing m the face f is to our right. Then, p_i must precede q_i, for $i = 1, 2$, since f is outside of $w|_{x_i}$. Assume, w.l.o.g., that p_1 precedes p_2. It follows from Observation 2.4 that l_2 crosses l_1 (see Figure 6(a)). Since $m \subset e|_{p_1,q_1} \cap e|_{p_2,q_2}$, the order of the four points is either p_1, p_2, q_1, q_2 or p_1, p_2, q_2, q_1. Let us consider these cases:

Case 1: Suppose q_1 precedes q_2 (see Figure 6(b)). Since $w_1|_{x_1}$ and $w_2|_{x_2}$ do not contain any original vertex, l_2 must cross l_1 (one more time) and r_1 (see Figure 6(c)), or r_2 must cross l_1 and r_1 (see Figure 6(d)). The first case yields an empty lens. In the second case, note that e'_1 must cross either l_2 (see Figure 6(e)) or r_2 (see Figure 6(f)), yielding four pairwise crossing edges.

Case 2: Suppose q_1 precedes q_2 (see Figure 6(g)). Then the edge e'_2 must cross e twice, creating an empty lens, or cross l_1, yielding four pairwise crossing edges (see Figure 6(h)).

Since all the cases imply forbidden configurations (an empty lens or four pairwise crossing edges) we conclude that f cannot be a possible $\mathcal{X}$-contributor through m to more than one wedge. $\square$

It follows from Observation 2.5 that a face f, such that $|f| \geq 7$, ends up with a charge of at least $|f| - 4 - |f|(\frac{1}{3} + \frac{1}{18}) > 0$. Likewise, k-quadrilaterals, for $k > 0$, and good pentagons and hexagons, end up with a non-negative charge, as their charge after charging the 0-triangles was at least $\frac{1}{3}$. Summing up the charge over all the wedges we have $\frac{2|E(G)|}{18} \leq 4n - 8$, hence $|E(G)| \leq 36n - 72$. $\square$



(a) l_2 must cross l_1. (b) q_1 precedes q_2. (c) l_1 and l_2 form an empty lens. (d) r_2 crosses l_1 and r_1.

(e) e_1' crosses l_2. (f) e_1' crosses r_2. (g) q_2 precedes q_1. (h) e_2' must cross l_1.

Figure 6: Illustrations for the proof of Observation 2.5

3. DISCUSSION

We have shown that the maximum number of edges in a topological graph on n vertices with no four pairwise crossing edge is $O(n)$. An interesting open problem is to determine the exact constant hiding in the O-notation. By noticing that it is impossible that all the faces incident to a vertex of G are 1-triangles (as done in [1]), one can reduce this constant to 28.8, but this is probably still not tight. The bound we found, combined with the analysis in [2] and [3], yields the following corollaries.

Corollary 1. *For any fixed integer $k > 4$, a simple topological graph on n vertices with no k pairwise crossing edges has $O(n \log^{2k-8} n)$ edges.*

Corollary 2. *For any fixed integer $k > 4$, a topological graph on n vertices with no k pairwise crossing edges has $O(n \log^{4k-16} n)$ edges.*

This improves the previous bounds by a factor of $\Theta(\log^2 n)$ and $\Theta(\log^4 n)$, respectively. However, the conjecture that $f_k(n) = O(n)$ for any fixed $k > 4$ remains open. It might be possible to settle this conjecture for $k = 5$ using our method, but it seems that for greater values, one should come up with new ideas.

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